The use of the logistic function for forecasting vertical movements of surface

Uporaba logistične funkcije pri napovedovanju vertikalnih premikov površine

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Abstract: We are familiar with a lot of different methods, which are used for forecasting vertical movements of a surface. These movements are caused by mining or building of an underground object. In this article we will in detail write about the logistic function and with its help we will describe time dependent vertical movements.

Izvleček: Poznamo veliko različnih metod, ki se uporabljajo za napovedovanje vertikalnih premikov površine, ki nastanejo zaradi rudarjenja ali izgradnje podzemnega objekta. V tem članku bomo podrobneje pisali o logistični funkciji in z njo opisali vertikalne premike v odvisnosti od časa.

Key words: logistic function, vertical movements, prognostic methods
Ključne besede: logistična funkcija, vertikalni pomiki, prognozne metode

Introduction

Experts are engaged in dynamics of subsidence of field and buildings which are results of mining from the very early stages of mining. Therefore in single areas different prognostic methods were developed to forecast movements of the field above the underground object.

In continuation it will be presented how we can forecast developing of vertical movements above the underground object with help of a logistic function. Basics of the logistic function will be described in the beginning (its structure and form). With help of this function we will then describe time dependent vertical movements. In the example, that will be presented, we will compare data achieved in the field above an underground object and calculated values achieved with help of the logistic function. In conclusion

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we analyse, if this method of forecasting vertical movements offers us good and useful results so we can continue using it in future.

Description of the logistic function

Structure of the logistic function

The logistic function consists of two parts: exponential and boundary exponential function. In the exponential function the growing is exponential which means that the growth rate is actually proportional to the size of the function value. In the second part of the logistic function the graph is approaching some fixed capacity, stated in advance. We achieve this by subtracting the exponential function from the fixed capacity. The function that we get in this way combines the first part of exponential growth, when the outputs are small, with the second part of exponential growth, when the outputs are approaching a certain limit.

Algebraic presentation

$$f(x) = \frac{a}{1 + b \cdot c^{-x}}$$

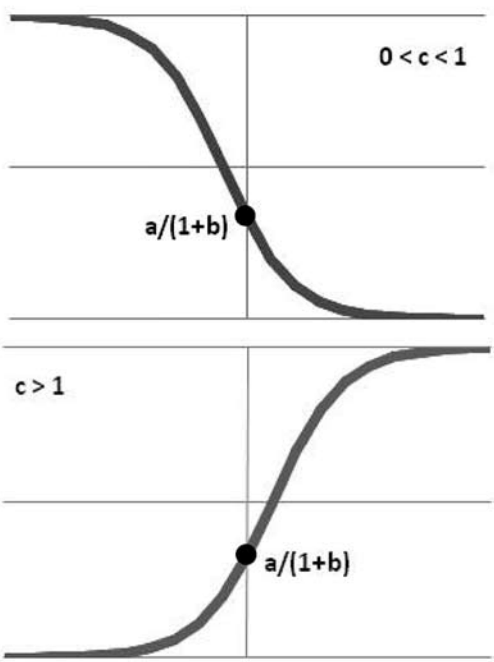

Figure 1. Influence of parameter ***c*** on logistic function
Slika 1. Vpliv parametra ***c*** na logistično funkcijo

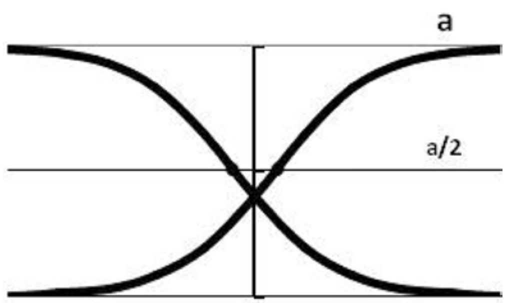

Figure 2. Inflection point
Slika 2. Prevojna točka

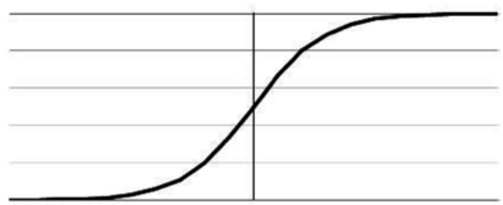

Figure 3. Shape of the logistic function
Slika 3. Oblika logistične funkcije

The algebra of the logistic function combines characteristics of exponential and power functions. This means that we are in the field where descriptions of complex problems are composed from more different functions.

To win a graph with the characteristic S-shape, we have to use three parameters of the logistic family which are connected to one another.

The parameters ***b*** and ***c*** are the base of the component of the exponential function $b \cdot c^{-x}$ and they define the point where the graph intercepts the y-axis. Base ***c*** is limited with positive values, so as long as $c > 1$, c^{x} grows and c^{-x} decays. Similarly if $0 < c < 1$, c^{x} decays and c^{-x} grows.

With other words this means: if c^{-x} grows ($0 < c < 1$), so does the denominator and a function as a whole is driven towards 0. If c^{-x} decays ($c > 1$), the denominator approaches 1 and the function as a whole converges towards the value of the numerator (parameter ***a***).

The rate at which a logistic function falls from or rises to its limiting value is completely determined by the exponential function in the denominator. More exactly, by the parameters ***b*** and ***c***.

The curve changes at its half from being convex to concave or the other way round, depending of the growing or decaying of the function.

The gradient change always appears on halfway of the logistic function. This point of critical change in the function's behaviour is called the inflection or gradation point.

By using this we can calculate the exact location of the point of inflection:

$$\frac{a}{1 + bc^{-x}} = \frac{a}{2}$$

$$1 + bc^{-x} = 2$$

$$bc^{-x} = 1$$

$$c^{-x} = b^{-1}$$

$$c^{x} = b$$

$$c^{x} = \log_{c} b$$

Graphic presentation
Logistic graphs follow a characteristic S-shape which is often called S-function or sigmoid function.

The S shape may either rise from the x-axis to the limiting value, or drop from the limiting value to the x-axis. The limiting value may change. The rate at which the curve travels between the two horizontal asymptotes may vary, but this basic sigmoid shape is found in all logistic graphs.

Logistic law and description of time depending vertical movements

$$u'_3(t) = au_3(t) - bu_3{}^2(t_0) \dots\dots\dots u_3(t=0) = u_{3,0} \tag{1}$$

u_3 .... vertical movement
$au_3(t)$..........growing part
$bu_3{}^2(t_0)$........ suffocating part, which prevents unlimited growth

Parameters ***a*** and ***b*** are constants which have an exactly fixed dimension:

$$\begin{aligned} a &\rightarrow [T^{-1}] \\ b &\rightarrow [M^{-1}T^{-1}] \end{aligned} \tag{2}$$

After integration of the equation (1) we get the logistic function (ČIBEJ, 1988):

$$u_3(t) = \frac{au_{3,0}}{bu_{3,0} + (a - bu_{3,0}) \cdot \exp(-a(t-t_0))} \tag{3}$$

Equation (3) is not practical for forecasting vertical movements because it contains too many reciprocally independent parameters. Therefore it would be right to assign the vertical movement in early stages $\boldsymbol{u_0}$ some minimal starting value, which would be considered in all observing points. As in case $u_{3,0} = 0$ also $u_3(t) = 0$ we will arrange to $\boldsymbol{u_{3,0}}$ a very small value, which will be different as zero.

So we adopt: $u_{3,0} = 0{,}2$ mm

Because of a shorter record we introduce factor λ with the following record:

$$\lambda = \frac{u_{3,max}}{u_{3,0}} = \frac{u_{3,max}}{0{,}2} \frac{[M]}{[M]} \tag{4}$$

We also define the maximum subsidence:

$$u_{3,max} = \lim_{t\to\infty} \frac{au_{3,0}}{bu_{3,0} + (a - bu_{3,0}) \cdot \exp(-a(t - t_0))} = \frac{au_{3,0}}{bu_{3,0}} = \frac{a}{b} \tag{5}$$

For the parameter ***b*** we introduce the following substitution:

$$u_{3,max} = \frac{a}{b} \tag{6}$$

$$b = \frac{a}{u_{3,max}} \tag{7}$$

Previous results of vertical movements on the surface, that we analyzed, have showed that one half of vertical movements is being developed in time t_c. Because the developing of vertical movements is the fastest in time t_c. After that time it begins slowing down. This characteristic of the process of subsidence we use to determinate the parameter ***a***.

From the mentioned it follows:

$$t = t_c$$
$$u_3(t_c) = \frac{1}{2} u_{3,max} \tag{8}$$

For the inflection point the logistic functions gets by taking into account equation (8), introduction of substitution (7) and introduction of the factor ***λ*** "λ" the following shape:

$$\frac{1}{2} u_{3,max} = \frac{a\frac{u_{3,max}}{\lambda}}{\frac{a}{u_{3,max}} \cdot \frac{u_{3,max}}{\lambda} + \left(a - \frac{a}{u_{3,max}} \cdot \frac{u_{3,max}}{\lambda}\right) \cdot \exp(-a(t_c - t_0))} \tag{9}$$

$$\frac{1}{2} = \frac{\frac{1}{\lambda}}{\frac{1}{\lambda} + \left(1 - \frac{1}{\lambda}\right) \cdot \exp(-a(t_c - t_0))} \tag{10}$$

From equation (10) we can express value ***a*** in dependence to λaccording to the following procedure:

$$\frac{1}{\lambda}+\left(1-\frac{1}{\lambda}\right)\cdot\exp\left(-a(t_c-t_0)\right)=2\cdot\frac{1}{\lambda}$$

$$\left(1-\frac{1}{\lambda}\right)\cdot\exp\left(-a(t_c-t_0)\right)=\frac{1}{\lambda}$$

$$\exp\left(-a(t_c-t_0)\right)=\left(\frac{1}{\lambda-1}\right) \tag{11}$$

$$a=-\frac{ln\left(\frac{1}{\lambda-1}\right)}{(t_c-t_0)}$$

After performing it we have all parameters of a S-shaped time diagram.

Example of using the logistic function

An example of comparison of observations with the logistic S function will be presented upon the example of point C20, where it means:

$u_{3,0}$.............. adopted starting value of movement
$u_{3,max}$........... maximum measured subsidence of a particular point
t_0 .... day, when the first measurement was made
t .... time of running measurement
t_c .... day when subsidence comes to half of its value
λ................. factor introduced for shorter recording
a................. constant
b................. constant

$u_{3,0}$ = 0,2 mm

$$u_{3,max}=-38\ mm\rightarrow\frac{u_{3,max}}{2}=-19\ mm$$

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$t_c=160\ dan^{-1}$ $t_c=160\ dan^{-1}$ …. day of the inflection point – we read this from the measuring table

$$\lambda=\frac{|u_{3,max}|}{u_{3,0}}=\frac{|-38|}{0{,}2}=190$$

$$a = -\frac{ln\left(\frac{1}{\lambda - 1}\right)}{(t_c - t_0)} = -\frac{ln\left(\frac{1}{190 - 1}\right)}{(160 - 0)} = 0{,}0327609$$

$$b = \frac{a}{|u_{3,max}|} = \frac{0{,}0327609}{|-38|} = 0{,}0008621$$

$$u_3(t) = \frac{a u_{3,0}}{b u_{3,0} + (a - b u_{3,0}) \cdot \exp\left(-a(t - t_0)\right)}$$

$$= \frac{0{,}0327609 \cdot 0{,}2}{0{,}0008621 \cdot 0{,}2 + (0{,}0327609 - 0{,}0008621 \cdot 0{,}2) \cdot \exp\left(-0{,}0327609 \cdot (t - t_0)\right)}$$

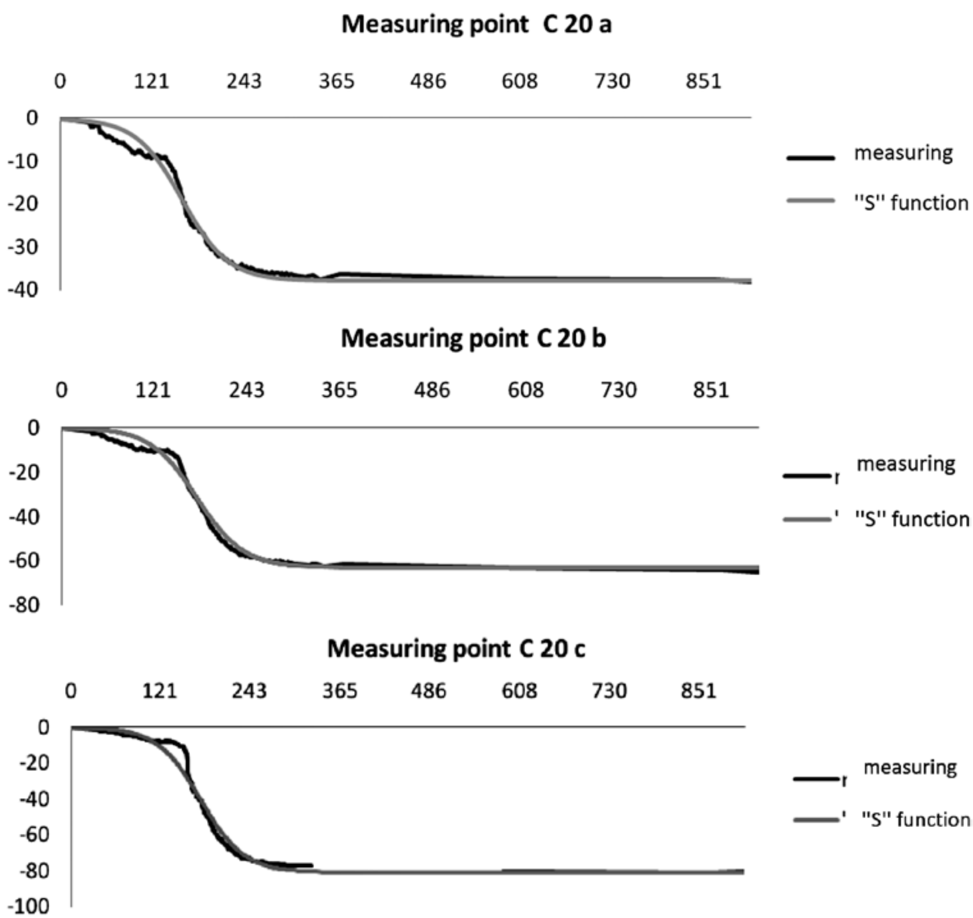

Figure 4. Comparing measured and calculated graph
Slika 4. Primerjava meritvenega in izračunanega grafa

CONCLUSIONS

As we see, the graphs of a single point are rather well agreeing with one another. From this fact we can presume, that this method can be used for forecasting of vertical movements above an underground object. But we mustn't forget that we get such results only by anticipating maximum subsidence and that could be a problem in some cases. We can therefore conclude that this method is based upon a theoretical part as well as inevitably upon an empirical part.

POVZETEK

Uporaba logistične funkcije pri napovedovanju vertikalnih premikov površine

Za primer in analizo uporabnosti logistične funkcije pri napovedovanju vertikalnih premikov, je bilo narejenih preko 20.000 meritev na več kot sto različnih točkah. Rezultati, ki smo jih dobili z logistično funkcijo (S – funkcija), se z dejanskimi meritvami zelo dobro ujemajo. Pri večini točk se empirična in teoretična krivulja skoraj popolnoma ujemata, kot je vidno tudi pri zgoraj navedenem primeru (slika 4). Največkrat pride do odstopanja v zaključnem delu grafa, ko se funkcija že umirja. Graf S - funkcije doseže mejno vrednost pred dejanskimi meritvami.

Slaba stran takšnega napovedovanja je, da moramo empirično ugotoviti maksimalni ugrezek. Če to vrednost poznamo, se rezultati zelo dobro ujemajo. Torej metoda temelji na teoretičnem in neizogibno tudi na empiričnem delu.

Vendar, tudi če naredimo veliko število meritev in lahko iz krivulje razberemo, da so vertikalni premiki površine minimalni, ne moremo z zagotovostjo trditi, da se ti premiki na meritvenem območju umirjajo. Zaradi tega so potrebne nadaljnje meritve in analize rezultatov.

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