The possibility of using homogeneous (projective) coordinates in 2D measurement exercises

Možnost uporabe homogenih (projektivnih) koordinat v dvodimenzionalnih merskih nalogah

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Abstract: This work’s intention is to present the basic characteristics of projective geometry and the use of homogeneous (projective) coordinates in two-dimensional (further denoted as 2D) measurement exercises. The concept of a projective plane originates from the Euclidean plane, assuming all our given points are ideal and lie upon an ideal line verging towards infinity. The term “ideal point” is taken to mean an intersection of all lines that are parallel in the finite space. By introducing these so-called ideal points and ideal lines, the calculations in 2D measurement exercises – ones that are usually carried out under the rules of Euclidean trigonometry – have been simplified, as the calculations of directional angles and lengths are no longer necessary. As a practical example of the use of projective coordinates, an intersection is presented, such that it can also be used for the Collins Method of Resection, seeing as it is based upon using said intersection twice.

Izvleček: Predstavljene so osnovne značilnosti projektivne geometrije in s tem uporaba homogenih (projektivnih) koordinat v dvodimenzionalnih merskih nalogah. Projektivno ravnino dobimo iz evklidske ravnine, če privzamemo točke in premico v neskončnosti na kateri ležijo vse točke v neskončnosti. Neskončna točka predstavlja presečišče vseh premic, ki so v končnosti med seboj vzporedne. Z uvedbo neskončnih točk in neskončne premice se izračunijo v dvodimenzionalnih merskih nalogah, ki se navadno vršijo po pravilih evklidske trigonometrije, poenostavijo saj računanje smernih kotov in dolžin ni potrebno. Kot praktični primer uporaba projektivnih koor-
Introduction

In 2D measurement exercises, all calculations are usually carried out under the rule of Euclidean geometry - where the points, lines and their relationships are defined differently than in a projective plane - in which the coordinates of an unknown point are established through the aid the calculation of so-called lengths and angles of our site. As a reaction to the latter, the article at hand is meant to present the basic relations between points and lines of the projective plane and depict their use in 2D measurement exercises. A practical example of using said projective coordinates would be an intersection, in which the coordinates of a new point are calculated from the measured angles of given points of an existing triangulation network. In what follows, the coordinates of a point and the formulas of lines on a Euclidean plane shall be marked using upper case, whereas the coordinates of points and formulas of lines on the projective plane will be denoted using lower case letters.

Point and Line

In projective geometry, a point is defined as a set of three coordinates that equal the set \((y \ x \ \omega)\) and therefore obviously also as an ordered set of three numbers \((y \ x \ \omega)\) - which do not all equal zero at the same time, since then \((\lambda y \ \lambda x \ \lambda \omega)\) would be the same point for any given \(\lambda \neq 0\).

For example, \((2 \ 3 \ 6)\) is our exemplary point, and \((\frac{1}{3} \ \frac{1}{2} \ 1)\) is another of the numerous ways to mark that exact same point, bearing in mind the principle that an unlimited number of sets of three numbers \((y \ x \ \omega)\) may correspond to each point, but only one point may correspond to each ordered set. Furthermore, from non-homogeneous coordinates of any given point, we beget an infinite number of sets of homogeneous coordinates of that same point:

\[
\lambda \neq 0 \Rightarrow \begin{pmatrix} Y \\ X \\ 1 \end{pmatrix} \rightarrow \lambda \begin{pmatrix} X \\ Y \\ \omega \end{pmatrix} = \begin{pmatrix} \lambda y \\ \lambda x \\ \lambda \omega \end{pmatrix}
\] (1)
And from homogeneous coordinates of that certain point, we can get one single ordered pair of numbers:

\[
\lambda \neq 0 \Rightarrow \lambda \left( \begin{array}{c}
\frac{\lambda y}{\lambda \omega} \\
\frac{\lambda x}{\lambda \omega} \\
\frac{\lambda \omega}{\lambda \omega}
\end{array} \right) = \left( \begin{array}{c}
\lambda y \\
\lambda x \\
\lambda \omega
\end{array} \right) = \lambda \left( \begin{array}{c}
\frac{\lambda y}{\lambda \omega} \\
\frac{\lambda x}{\lambda \omega} \\
\frac{\lambda \omega}{\lambda \omega}
\end{array} \right) = \lambda \left( \begin{array}{c}
\frac{\lambda y}{\lambda \omega} \\
\frac{\lambda x}{\lambda \omega} \\
\frac{\lambda \omega}{\lambda \omega}
\end{array} \right) = \left( \begin{array}{c}
y' \\
x' \\
\omega'
\end{array} \right)
\]

A line is defined in almost the same way as a point, the one difference being that the line is treated as a set of all points that equal the set of three \((v\ u\ w)\). For example, if \((3\ 2\ -2)\) defines a line, then \((-3/2\ -1\ 1)\) is the notation for that same line. In other words, a line is an ordered set of three numbers, denoted as \((v\ u\ w)\), which do not equal zero all at the same time, in which case \((\mu v\ \mu u\ \mu w)\) would be the same line for any given \(\mu \neq 0\)\(^{[1]}\).

**Incidence relation**

Given point \(P\) and line \(l\) - where \(P\) and \(l\) are short for \((y\ x\ \omega)\) and \((v\ u\ w)\), respectively - we say that the two are incident to one another (incidence relation being descriptive of the relation simply summed up as the point lying on the line i.e. the line going “through” the point) in the case, the following is true: \(\{P \cdot l\} = \{l \cdot P\} = vy + xu + ow = vy + ux + w\omega = 0\)\(^{[2]}\).

**The principle of duality**

The axiom of geometry says that there is exactly one line that is in incidence with two different points \(P_1\) and \(P_2\), i.e. exactly one line goes through both different points. If we exchange the term “line” with the term “point” in the above axiom, we get a geometrical theorem that states there is exactly one point that is in incidence with two different lines \(l_1\) and \(l_2\) - in other words, two different lines intersect at exactly one point in space. The afore-mentioned axiom and theorem are said to be mutually dual, i.e. points and lines are mutually dual spatial elements, while the “running” and intersecting of lines are mutually dual operations\(^{[3]}\).

For different points \(P_1(y_1\ x_1\ \omega_1)\) and \(P_2(y_2\ x_2\ \omega_2)\); \(y_1 \neq \lambda \cdot y_2, x_1 \neq \lambda \cdot x_2, \omega_1 \neq \lambda \cdot \omega_2\) there is always an ordered set of three elements, \((v\ u\ w)\), for which the following need most definitely apply:

\[
\begin{align*}
y_1v + xu + ow &= 0 \\
y_2v + x_2u + ow &= 0
\end{align*}
\]

\(^{[1]}\) RMZ-M&G 2008, 55
And for two different lines, in our particular example \( l_1 \) \((v_1 u_1 w_1)\) and \( l_2 \) \((v_2 u_2 w_2)\); \( v_1 \neq \mu \cdot v_2, u_1 \neq \mu \cdot u_2, w_1 \neq \mu \cdot w_2 \); there is always an ordered set of three elements, \((v x \omega)\), for which the following need apply:

\[
\begin{align*}
v_1y + u_1x + w_1 \omega &= 0 \\
v_2y + u_2x + w_2 \omega &= 0
\end{align*}
\]

Thus, homogeneous systems of equations are combined with the following expression:

\[
\begin{pmatrix}
{P} \\
{l_1} \\
{l_2}
\end{pmatrix} =
\begin{pmatrix}
{v} \\
{u} \\
{w}
\end{pmatrix}_1 \\
\begin{pmatrix}
v \\
u \\
w
\end{pmatrix}_2 = 0 \Rightarrow
\begin{pmatrix}
{P} \\
{l_1} \\
{l_2}
\end{pmatrix} =
\begin{pmatrix}
y \\
x \\
\omega
\end{pmatrix}_1 =
\begin{pmatrix}
y \\
x \\
\omega
\end{pmatrix}_2
\]

Then we are able to beget the formula of the point or line:

\[
\begin{pmatrix}
y \\
v \\
u \\
v_1 \\
v_2 \\
x_1 \\
x_2 \\
y_1 \\
y_2 \\
\omega_1 \\
\omega_2
\end{pmatrix} =
\begin{pmatrix}
y \\
v \\
u \\
x_1 \\
x_2 \\
y_1 \\
y_2 \\
\omega_1 \\
\omega_2
\end{pmatrix} +
\begin{pmatrix}
y \\
v \\
u \\
x_1 \\
x_2 \\
y_1 \\
y_2 \\
\omega_1 \\
\omega_2
\end{pmatrix} = 0
\]

Each set of three \(\begin{pmatrix}y \\v \\u \end{pmatrix}\) therefore represents a solution to the system under question.

The coordinates of the point or line follow:

\[
\begin{pmatrix}
y \\
v \\
u \\
x_1 \\
x_2 \\
y_1 \\
y_2 \\
\omega_1 \\
\omega_2
\end{pmatrix} =
\begin{pmatrix}
y \\
v \\
u \\
x_1 \\
x_2 \\
y_1 \\
y_2 \\
\omega_1 \\
\omega_2
\end{pmatrix}
\]

A point or a line pertaining to a projective plane can be multiplied by any number \(\lambda\) or \(\mu\), as long as the value is not equal to zero, which then gives us the actual coordinates of this same point or line:

\[
\begin{pmatrix}
y \\
v \\
u \\
x_1 \\
x_2 \\
y_1 \\
y_2 \\
\omega_1 \\
\omega_2
\end{pmatrix} = \begin{pmatrix} \lambda \\ u \\
\end{pmatrix}
\]
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For a point, this would be:

\[(y \ x \ \omega) = \mu \begin{pmatrix} w_1 & v_1 & u_1 \\ w_2 & v_2 & u_2 \end{pmatrix} \]

(9)

And for a line:

\[(v \ u \ \omega) = \lambda \begin{pmatrix} x_1 & \omega_1 & y_1 \\ x_2 & \omega_2 & y_2 \end{pmatrix} \]

(10)

**The Ideal Point and Line**

The parallels (coefficient of site \( k = k_1 = k_2 \)) \( l_1 \) and \( l_2 \) with formulas \( Y = k_1X + n_1 \) and \( Y = k_2X + n_2 \) are given. For \( X \), we enter values \( X = \frac{x}{\omega} \), and for \( Y \) values \( Y = \frac{y}{\omega} \), then multiplying the equations with \( \omega \), which in turn gives us the formulas of lines using homogeneous coordinates:

\[y = kx + n_1\omega \quad \text{and} \quad y = kx + n_2\omega \]

(11)

Considering that the lines are parallel, we are interested in the set of three, \((v \ u \ w)\), such that it must correspond to both formulas.

By subtracting the equations we get: \( \omega(n_1 - n_2) = 0 \).

As \( n_1 \neq n_2 \), then \( \omega = 0 \) and the equations \( y = kx + n_1\omega \) and \( y = kx + n_2\omega \) are reduced into \( y = xk \).

Since we are dealing with homogeneous coordinates, we can say that \( x = 1 \). From this, we come to the conclusion that \( y = k \). Thus, we obtain a set of three, \((k \ 1 \ 0)\), which does indeed correspond to both equations.

If the lines \( l_1 \) and \( l_2 \) are parallel to the \( y \)-axis, then the formulas of the lines in homogeneous coordinates have the form \( x = \omega x_1 \) and \( x = \omega x_2 \). In this case, the set of three \((0 \ 1 \ 0)\) corresponds to both formulas.

To summarize, the set of three, \((k \ 1 \ 0)\), corresponds to a formula of the forms \( y = kx + n_1\omega \) and \( y = kx + n_2\omega \) only when \( k = k_1 = k_2 \), or when the lines are parallel and the coefficient of site equals \( k \). The set of three \((0 \ 1 \ 0)\) corresponds to all formulas of the form \( x = \omega x_1 \) that describe the parallels of the \( y \)-axis.

A bouquet of parallel lines (all parallel lines are of the same class and form organised heaps of parallel lines denoted as “bouquets”) defines a point \( P_\infty \) in projective plane that has been defined as an ideal point. The bouquet \( P_\infty \) consists of all lines that are parallel to
a certain line $l$. The equation pertaining to line $l$ is $Y = kX + n$, or $X = X_p$ if it is parallel to the $y$ rather than the $x$-axis.

The line $l$ belongs to bouquet $P_\infty$ exactly when the set of three $(k \ 1 \ 0)$ corresponds to the equation of line $l$ in its homogeneous coordinates, and to pencil $P_\infty$ exactly when what corresponds to this equation is this set of three: $(0 \ 1 \ 0)$. Consequentially, we can have the set of three $(k \ 1 \ 0)$ in the former case, and the set of three $(0 \ 1 \ 0)$ in the latter case for homogeneous coordinates of the ideal point $P_\infty$.

Since we may multiply homogeneous coordinates with any number that is different than zero, we may say that the set of three $(y \ x \ 0)$ represents the homogeneous coordinates of one ideal point, where $y$ and $x$ are any given elements different from zero. In this way, we have adjusted our homogeneous coordinates $(y \ x \ \omega)$ so that they befit each and every point of our projective plane. The point with such coordinates also lies in the Euclidean plane if $\omega \neq 0$, and is an ideal point when $\omega = 0$.[4]

Two ideal points define the ideal line $l_\infty$:

$$\begin{vmatrix} v_\infty & u_\infty & w_\infty \\ y_1 & x_1 & 0 \\ y_2 & x_2 & 0 \end{vmatrix} = 0$$

The solution of the system are the very coordinates of our ideal line $l_\infty$:

$$\left(\begin{array}{c} v_\infty \\ u_\infty \\ w_\infty \end{array}\right) = \mu \left(\begin{array}{c} x_1 \\ x_2 \\ 0 \end{array}\right) = \mu \left(\begin{array}{c} y_1 \\ y_2 \\ 0 \end{array}\right) = \left(\begin{array}{c} y_1/x_1 \\ y_2/x_2 \end{array}\right)$$

or, if:

$$\left(\begin{array}{c} v_\infty \\ u_\infty \\ w_\infty \end{array}\right) = \left(\begin{array}{c} y_1 \\ y_2 \\ 0 \end{array}\right) = \left(\begin{array}{c} 1 \\ y_1/x_1 \\ y_2/x_2 \end{array}\right) \Rightarrow \left(\begin{array}{c} v_\infty \\ u_\infty \\ w_\infty \end{array}\right) = (0 \ 0 \ 1)$$

The set of three $(0 \ 0 \ 1)$ represents the coordinates of an ideal line such that all ideal points lie on it.

**The ideal line and angle of site**

An ideal point represents the intersection of a group of all lines that are parallel to one another in finite space. A specific ideal point upon an ideal line belongs to each group of parallel lines that in finite space represent a so-called »angle of site« between the lines of a certain class and the positive end of the x-axis.
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The coordinates of the line that goes through the points \( P_1(y_1, x_1, \omega_1) \) and \( P_2(y_1 + d \cos \varphi, x_1 + d \sin \varphi, \omega_2) \), where \( \varphi \) is the so-called angle of site are enclosed within the line \( P_1P_2 \) and the positive side of \( x \)-axis, and \( d \) represents the distance between the points \( P_1 \) and \( P_2 \), which would be:

\[
(v \ u \ w) = \mu (x_1(\omega_2 - \omega_1) - d \cos \varphi \omega_1, y_1(\omega_1 - \omega_2) + d \sin \varphi \omega_2, d(y_1 \cos \varphi - x_1 \sin \varphi)) \quad (15)
\]

If:

\[
\omega_2 = \omega_1 = 1 \Rightarrow (v \ u \ w) = \mu (-d \cos \varphi, d \sin \varphi, d(y_1 \cos \varphi - x_1 \sin \varphi)) \quad (16)
\]

or, when:

\[
(\omega_2 = \omega_1 = 1 \land d \neq 0 \land d \neq \infty \land \mu = \frac{1}{d}) \Rightarrow (v \ u \ w) = (- \cos \varphi \sin \varphi, y_1 \cos \varphi - x_1 \sin \varphi)) \quad (17)
\]

The set \((v \ u \ w) = (- \cos \varphi \sin \varphi, y_1 \cos \varphi - x_1 \sin \varphi))\) represents the coordinates of the line that is notated using polar coordinates.

The intersection of the line given with polar coordinates and the ideal line \((0 \ 0 \ 1)\) is the ideal line, now denoted using polar coordinates:

\[
(y_\infty \ x_\infty \ \omega_\infty) = \lambda \left[ \begin{array}{ccc}
\sin \varphi & y_1 \cos \varphi - x_1 \sin \varphi & y_1 \cos \varphi - x_1 \sin \varphi - \cos \varphi \\
0 & 1 & 0 \\
\cos \varphi & \sin \varphi & 0
\end{array} \right] \quad (18)
\]

or, if:

\[
\lambda = 1 \Rightarrow (y_\infty \ x_\infty \ \omega_\infty) = (\sin \varphi, \cos \varphi, 0) \quad (19)
\]

The notation \((y_\infty \ x_\infty \ \omega_\infty) = (\sin \varphi, \cos \varphi, 0)\) at the same time also represents standardised coordinates of said ideal line, for which the following is valid:

\[
\begin{pmatrix}
y_\infty \\
x_\infty \\
\omega_\infty
\end{pmatrix} = \begin{pmatrix}
\frac{y_\infty}{\sqrt{y_\infty^2 + x_\infty^2}} \\
\frac{x_\infty}{\sqrt{y_\infty^2 + x_\infty^2}} \\
0
\end{pmatrix} = 1 \quad (20)
\]

where:

\[
\varphi = \tan^{-1} \left( \frac{\sin \varphi}{\cos \varphi} \right) = \frac{\sqrt{y_\infty^2 + x_\infty^2}}{x_\infty} = \tan^{-1} \left( \frac{y_\infty}{x_\infty} \right) \quad (21)
\]
Should the lines $l_1$ in $l_2$ enclose the angle $\alpha$, and the line $l_i$ is defined through the ideal point $(\sin \varphi_1 \cos \varphi_1, 0)$, then the ideal point of line $l_2$ is defined as $(\sin (\varphi_1 + \alpha) \cos (\varphi_1 + \alpha), 0)$, where $\varphi_1 + \alpha$ is the angle of site of the line $l_2$:

**Figure 1.** The angle between our two lines and angles of site

**Slika 1.** Kot med prenicama in smerna kota

**Intersection**

An intersection is both a measurement and calculation method, with the aid of which the coordinates of a new point can be calculated from measured angles or (outer) directions upon given points of the existing triangulation network. A given new point is determined as an intersection of several outer directions that are controllably oriented at each standpoint.

The coordinates of points $L(Y_L, X_L)$ and $R(Y_R, X_R)$ are given.

**Observation:**

We are observing the direction from two given points ($L$ and $R$) towards the new point $M$. Angle $\alpha$ is observed from the point $L$ between points $M$-left and $R$-right, and angle $\beta$ from point $R$ between points $L$-left and $M$-right.

Based on the given and observed information, we must establish the coordinates of the unknown point $M$, which would translate into us looking for $X_M$ and $Y_M$:
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The connecting line $\overline{LM}$ denotes the line left, $\overline{RM}$ the line right, and $\overline{LR}$ the line left-right. The lines left and left-right are bisected via one another in point $L$ and enclose the angle $\alpha$, while the lines right and left-right do the same in point $R$, therefore enclosing the angle $\beta$.

By establishing the ideal point $L_\infty$ of the line left and the ideal point $R_\infty$ of the line right, the lines left and right have been accurately established. Thus, we have also established their intersection $M$, which represents the coordinates of the point we are seeking.

$L_\infty$ denotes the ideal point on the line left, $R_\infty$ the ideal point on the line right and $LR_\infty$ the ideal point on the line left-right. Points $L_\infty$ and $R_\infty$ are obtained with the help of the ideal point $LR_\infty$, which is established through the bisection of line left-right with the ideal line. When the point $LR_\infty$ is standardised (transformed into a format such as $(\sin \varphi \ \cos \varphi \ 0)$), we can, with the help of the latter, as well as with the help of angles $\alpha$ and $\beta$, determine the points $L_\infty$ and $R_\infty$ on the line $l_\infty$.

1. Establishing the line left-right
- The coordinates of point $L$ are $(y_L = Y_L \ x_L = X_L \ 1)$
- The coordinates of point $R$ are $(y_R = Y_R \ = Y_L + X_L + d \sin \nu_L \ x_R = X_R = X_L + d \cos \nu_L \ 1)$

Therefore, the following is true:

$$\begin{vmatrix}
\nu_{LR} & u_{LR} & w_{LR} \\
Y_L & X_L & 1 \\
Y_L + d \cdot \sin \nu_L & X_L + d \cdot \cos \nu_L & 1
\end{vmatrix} = 0$$  \ \ \ \ (22)
The formula of the line \textit{left-right} is:

\begin{equation}
V_{LR} \begin{vmatrix}
X_L & 1 \\
X_L + d \cdot \cos \psi_R & 1
\end{vmatrix} - U_{LR} \begin{vmatrix}
Y_L & 1 \\
Y_L + d \cdot \sin \psi_R & 1
\end{vmatrix} + W_{LR} \begin{vmatrix}
Y_L & 1 \\
Y_L + d \cdot \sin \psi_R & 1
\end{vmatrix} = 0
\end{equation}

(23)

or, its coordinates:

\begin{equation}
V_{LR} = \begin{vmatrix}
X_L & 1 \\
X_L + d \cos \psi_R & 1
\end{vmatrix}
\end{equation}

(24)

\begin{equation}
U_{LR} = \begin{vmatrix}
1 & Y_L \\
Y_L + d \sin \psi_R & 1
\end{vmatrix}
\end{equation}

(25)

\begin{equation}
W_{LR} = \begin{vmatrix}
Y_L & X_L \\
Y_L + d \cdot \sin \psi_R & X_L + d \cdot \cos \psi_R
\end{vmatrix}
= \begin{vmatrix}
Y_L & X_L \\
Y_L + d \cdot \sin \psi_R & X_L + d \cdot \cos \psi_R
\end{vmatrix}
\end{equation}

(26)

2. \textit{Establishing the ideal point on the line left-right}

The line \textit{left-right} is bisected with the ideal point (0 0 1):

\begin{equation}
\begin{vmatrix}
y_{LR} \\
X_L \\
X_L + d \cdot \cos \psi_R
\end{vmatrix}
\begin{vmatrix}
x_{LR} \\
Y_L \\
Y_L + d \cdot \sin \psi_R
\end{vmatrix}
\begin{vmatrix}
\omega_{LR} \\
1 \\
Y_L + d \cdot \sin \psi_R
\end{vmatrix}
= 0
\end{equation}

(27)

The coordinates of the ideal point are denoted as follows:

\begin{equation}
y_{LR} = \begin{vmatrix}
0 \\
1 \\
1 Y_L + d \cdot \sin \psi_R \\
Y_L + d \cdot \sin \psi_R & d \cdot \cos \psi
\end{vmatrix}
= \begin{vmatrix}
Y_L \\
1 Y_L + d \cdot \sin \psi_R
\end{vmatrix}
\end{equation}

(28)

\begin{equation}
x_{LR} = \begin{vmatrix}
1 \\
0 \\
0 \\
0 & 1
\end{vmatrix}
= \begin{vmatrix}
X_L \\
X_L + d \cdot \cos \psi_R
\end{vmatrix}
\end{equation}

(29)

\begin{equation}
\omega_{LR} = \begin{vmatrix}
0 \\
0 \\
1 \\
0 & 1
\end{vmatrix}
\end{equation}

(30)
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The point $LR_\infty = \lambda(-d \sin \nu_L^R - d \cos \nu_L^R 0)$ is standardised, which means that it is multiplied by factor:

$$\lambda = \frac{1}{\sqrt{y_{LR_\infty}^2 + x_{LR_\infty}^2 + \omega_{LR_\infty}^2}}$$

Consequentially, we transform the notation $LR_\infty = \lambda(-d \sin \nu_L^R - d \cos \nu_L^R 0)$ into $LR_\infty = \lambda(\pm \sin \nu_L^R \pm \cos \nu_L^R 0)$.

$$LR_\infty = \lambda \begin{pmatrix} -d \cdot \sin \nu_L^R & -d \cdot \cos \nu_L^R & 0 \end{pmatrix}$$ (31)

$$LR_\infty = \lambda \begin{pmatrix} -d \cdot \sin \nu_L^R & -d \cdot \cos \nu_L^R & 0 \end{pmatrix}$$ (32)

$$LR_\infty = \lambda \begin{pmatrix} -d \cdot \sin \nu_L^R & -d \cdot \cos \nu_L^R & 0 \end{pmatrix}$$ (33)

$$LR_\infty = \lambda \begin{pmatrix} -d \cdot \sin \nu_L^R & -d \cdot \cos \nu_L^R & 0 \end{pmatrix}$$ (34)

So, finally, the standardised ideal point is:

$$LR_\infty = \begin{pmatrix} \mp \sin \nu_L^R & \mp \cos \nu_L^R & 0 \end{pmatrix}$$ (36)

or:

$$LR_\infty = \begin{pmatrix} \mp \sin \left(\nu_L^R + k\pi\right) & \mp \cos \left(\nu_L^R + k\pi\right) & 0 \end{pmatrix}$$ (37)

3. Establishing the ideal points on the lines left and right

(a) Ideal point $L_\infty$ of the line left

The line left-right encloses, along with the positive side of $x$-axis, the angle $\nu_L^R$, and with the line left, the angle $\alpha$. The angle of site of the line left is $\nu_L^M = \nu_L^R - \alpha$ and the ideal line is of the form:

$$L_\infty = \begin{pmatrix} \sin (\nu_L^R - \alpha) \cos (\nu_L^R - \alpha) & 0 \end{pmatrix}$$ (38)

(b) Ideal point $R_\infty$ of the line right

The coefficient of site of the line right is $\nu_L^M = \nu_L^R + \beta$ and the ideal line is of the form:

$$R_\infty = \begin{pmatrix} \sin (\nu_L^R + \beta) \cos (\nu_L^R + \beta) & 0 \end{pmatrix}$$ (39)
4. Establishing the lines left and right

(a) Establishing the line left:

The line left goes through the points \( L_\infty \) and \( L \):

\[
\begin{vmatrix}
  v_{left} \\
  \sin v^L_L \cdot \cos \alpha - \cos v^R_L \cdot \sin \alpha \\
  Y_{left} \\
  \cos v^R_L \cdot \cos \alpha + \sin v^R_L \cdot \sin \alpha \\
  X_{left}
\end{vmatrix}
\begin{vmatrix}
  u_{left} \\
  0 \\
  1
\end{vmatrix}
= 0
\]

(40)

\[
v_{left} = \frac{\cos v^R_L \cdot \cos \alpha + \sin v^R_L \cdot \sin \alpha}{X_{left}}
= \cos (v^R_L - \alpha)
\]

(41)

\[
u_{left} = \frac{0 \sin v^R_L \cdot \cos \alpha - \cos v^R_L \cdot \sin \alpha}{1}
= -\sin (v^R_L - \alpha)
\]

(42)

\[
w_{left} = \frac{\sin v^R_L \cdot \cos \alpha - \cos v^R_L \cdot \sin \alpha \cos v^R_L \cdot \cos \alpha + \sin v^R_L \cdot \sin \alpha}{Y_{left}}
= -Y_{left} \cdot \cos (v^R_L - \alpha) - X_{left} \cdot \sin (v^R_L - \alpha)
\]

(43)

Coordinates of the line left:

\[
(v_{left} \ u_{left} \ w_{left}) = \left( \cos (v^R_L - \alpha) - \sin (v^R_L - \alpha) - Y_{left} \cdot \cos (v^R_L - \alpha) - X_{left} \cdot \sin (v^R_L - \alpha) \right)
\]

(44)

(b) Establishing the line right

The line right is defined by the points \( R \) and \( R_\infty \):

\[
\begin{vmatrix}
  v_{right} \\
  \sin v^L_R \cdot \cos \beta + \cos v^L_R \cdot \sin \beta \\
  Y_{right} \\
  \cos v^L_R \cdot \cos \beta - \sin v^L_R \cdot \sin \beta \\
  X_{right}
\end{vmatrix}
\begin{vmatrix}
  u_{right} \\
  0 \\
  1
\end{vmatrix}
= 0
\]

(45)

\[
v_{right} = \frac{\cos v^L_R \cdot \cos \beta - \sin v^L_R \cdot \sin \beta}{X_{right}}
= \cos (v^L_R + \beta)
\]

(46)

\[
u_{right} = \frac{0 \sin v^L_R \cdot \cos \beta + \cos v^L_R \cdot \sin \beta}{1}
= -\sin (v^L_R + \beta)
\]

(47)

\[
w_{right} = \frac{\sin v^L_R \cdot \cos \beta + \cos v^L_R \cdot \sin \beta \cos v^L_R \cdot \cos \beta - \sin v^L_R \cdot \sin \beta}{Y_{right}}
= -Y_{right} \cdot \cos (v^L_R + \beta) + X_{right} \cdot \sin (v^L_R - \beta)
\]

(48)
The possibility of using homogeneous (projective) coordinates...

Coordinates of the line right:

\[
(v_{\text{right}}, u_{\text{right}}, w_{\text{right}}) = \left( \cos\left(\frac{v_r}{R} + \beta\right) - \sin\left(\frac{v_r}{R} + \beta\right), -Y_{\text{right}}, \cos\left(\frac{v_r}{R} + \beta\right) + X_{\text{right}} \cdot \sin\left(\frac{v_r}{R} + \beta\right) \right)
\]  

(49)

5. Establishing the unknown point \( M \)

The point \( M \) is defined as the intersection of the lines \( \overline{LM} = L_\infty L \) and \( \overline{RM} = R_\infty R \):

\[
\begin{vmatrix}
    y_M & x_M & \omega_M \\
    v_{\text{left}} & u_{\text{left}} & w_{\text{left}} \\
    v_{\text{right}} & u_{\text{right}} & w_{\text{right}}
\end{vmatrix} = 0
\]

(50)

\[
M = \left( y_M, x_M, \omega_M \right) = \left( \begin{array}{ccc|ccc}
    v_{\text{left}} & w_{\text{left}} & & w_{\text{left}} & v_{\text{left}} & & \\
    u_{\text{left}} & w_{\text{left}} & & w_{\text{left}} & v_{\text{left}} & & \\
    v_{\text{right}} & u_{\text{right}} & & v_{\text{right}} & u_{\text{right}} & &
\end{array} \right)^{-1}
\]

(51)

\[\text{Figure 3. Intersection with projective coordinates}
\]

\[\text{Slika 3. Zunanji urez s projektivnimi koordinatami}\]
Conclusions

In projective geometry, the lines have their own coordinates inasmuch as points do. A point is defined as an ordered set of three numbers \((y \ x \ \omega)\), which are not allowed to simultaneously take on the value of zero, seeing as the latter would make \((\lambda y \ \lambda x \ \lambda \omega)\) the same point for any given \(\lambda \neq 0\). We can obtain an infinite number of ordered sets of coordinates of a given point from the non-homogeneous coordinates of that same point, just as well as we can obtain only one ordered pair of numbers from homogeneous coordinates.

A line, much like a point, is an ordered set of three numbers, \((v \ u \ w)\), which must not all equal zero at the same time, seeing as that would again make \((\mu v \ \mu u \ \mu w)\) the same line for any given \(\mu \neq 0\). The set \((v \ u \ w) = (x_1-x_2 \ y_1-y_2 \ y_1x_2-y_2x_1)\) represents the coordinates of the line that are denoted using rectangular coordinates, whilst the set \((v \ u \ w) = (-\cos \phi \ \sin \phi \ y \ \cos \phi - x \ \sin \phi)\) represents the coordinates of the line that is denoted using polar coordinates.

By using projective coordinates, the establishment of the intersection \(R(Y \ X)\) of two lines – the line \(l_1\), defined by the points \(A_1(Y_1 \ X_1)\) and \(A_2(Y_2 \ X_2)\), and the line \(l_2\), on which the points \(B_1(Y_1 \ X_1)\) and \(B_2(Y_2 \ X_2)\) lie – is very simple, and is defined using the help of nine second order determinants:

First, the coordinates of lines \(l_1\) and \(l_2\) are established, for which six of secondary-order determinants need to be calculated:

\[
\lambda = 1 \Rightarrow l_1 = (v_1 \ u_1 \ w_1) = \begin{vmatrix} X_1 & 1 & Y_1 \\ X_2 & 1 & Y_2 \end{vmatrix} \begin{vmatrix} Y_1 & X_1 \\ Y_2 & X_2 \end{vmatrix}
\]

\[
\lambda = 1 \Rightarrow l_2 = (v_2 \ u_2 \ w_2) = \begin{vmatrix} X_1 & 1 & Y_1 \\ X_2 & 1 & Y_2 \end{vmatrix} \begin{vmatrix} Y_1 & X_1 \\ Y_2 & X_2 \end{vmatrix}
\]

The intersection \(R\), as the intersection of lines \(l_1\) and \(l_2\), is established with the calculation of another three secondary-order determinants:

\[
\mu = 1 \Rightarrow R = (y \ x \ \omega) = \begin{vmatrix} u_1 & w_1 & v_1 \\ u_2 & w_2 & v_2 \end{vmatrix} \begin{vmatrix} v_1 & u_1 \\ v_2 & u_2 \end{vmatrix}
\]

\[
R = (y \ x \ \omega) = \begin{pmatrix} y \\ \omega \ \omega \end{pmatrix} = (Y \ X \ 1) = (Y' \ X')
\]
The possibility of using homogeneous (projective) coordinates...

Two different ideal points define the ideal line (0 0 1), on which all ideal points lie. The notation \((\sin \varphi \cos \varphi 0)\) represents these coordinates of our ideal point, from which we can find the angle which a given line encloses in partnership with the positive side of the \(x\)-axis.

If the line \(l_1\) and the positive side of the first axis enclose the angle \(\varphi_1\), and both \(l_1\) and \(l_2\) enclose the angle \(\alpha\), then the ideal point of the line \(l_2\) can be determined as \((\sin(\varphi_1 \pm \alpha) \cos(\varphi_1 \pm \alpha) 0)\), where \(\varphi_2 = \varphi_1 \pm \alpha\) is the angle of site of the line \(l_2\).

Via the introduction of ideal points and an ideal line, we are able to avoid the process of calculation of lengths and angles of site in 2D measurement exercises (the practical example here being the one depicted using intersections). The coordinates of an unknown point may always be established as the intersection of two lines, defined by the given and ideal points.

**Povzetek**

**Možnost uporabe homogenih (projektivnih) koordinat v dvodimenzionalnih merških nalogah**

V projektivni geometriji imajo poleg točk tudi premice koordinate.

Točka je definirana kot urejena trojka števil \((v x \omega)\), ki niso vse hkrati enake nič, s tem da je \((\lambda v \lambda x \lambda \omega)\) ista točka za katerikoli \(\lambda \neq 0\). Iz nehomogenih koordinat neke točke dobimo neskončno urejenih trojk homogenih koordinat iste točke, iz homogenih koordinat neke točke pa lahko dobimo eno samo urejeno dvojico števil.

Premica je urejena trojka števil \((v u w)\), ki niso vse hkrati enake nič, s tem da je \((\mu v \mu u \mu w)\) ista premica za katerikoli \(\mu \neq 0\). Trojka \((v u w) = (x_1 - x_2, y_1 - y_2, x_1 y_2 - y_1 x_2)\) predstavlja koordinate premice zapisane s pravokotnimi koordinatami, trojka \((v u w) = (- \cos \varphi \sin \varphi y_1 \cos \varphi - x_1 \sin \varphi)\) pa koordinate premice zapisane s polarnimi koordinatami.

Z uporabo projektivnih koordinat je določitev presečišča \(R(Y X)\) dveh premic, premice \(l_1\), ki ju določata točki \(A_1(Y_1 X_1)\) in \(A_2(Y_2 X_2)\) ter premice \(l_2\) na kateri ležita točki \(B_1(Y_1 X_1)\) in \(B_2(Y_2 X_2)\) zelo enostavna, saj se presečišče določi s pomočjo devetih determinant drugega reda:

Najprej se določijo koordinate premic \(l_1\) in \(l_2\) za kar je potrebno rešiti šest determinant drugega reda:

\(RMZ-M&G\ 2008, 55\)
Presečišče $R$, kot presek premic $l_1$ in $l_2$ pa je odrejeno z rešitvijo še treh determinant drugega reda:

$$\lambda = 1 \Rightarrow l_1 = (v_1 \quad u_1 \quad w_1) = \begin{pmatrix} X_1 & Y_1 & X_1 \\ X_2 & Y_2 & X_2 \end{pmatrix}$$

$$\lambda = 1 \Rightarrow l_2 = (v_2 \quad u_2 \quad w_2) = \begin{pmatrix} X_1 & Y_1 & X_1 \\ X_2 & Y_2 & X_2 \end{pmatrix}$$

Projektivno ravnino dobimo iz evklidske ravnine, če privzamemo točke in premico v neskončnosti. Točka s koordinatami $(y \ x \ \omega)$ leži v evklidski ravnini, če je $\omega \neq 0$ in je točka v neskončnosti, če je $\omega = 0$.

Dve različni neskončni točki določata premico v neskončnosti $(0 \ 0 \ 1)$ na kateri ležijo vse točke v neskončnosti. Zapis $(\sin \varphi \ \cos \varphi \ 0)$ predstavlja koordinate neskončne točke iz katerih razberemo kot, ki ga neka premica oklepa s pozitivnim delom abscisne osi.

Če oklepa premica $l_1$ s pozitivnim delom prve osi kot $\varphi_1$ s premico $l_2$ pa kot $\alpha$ lahko neskončno točko premice $l_2$ določimo kot $(\sin (\varphi_1 \pm \alpha) \ \cos (\varphi_1 \pm \alpha) \ 0)$ pri čemer je $\varphi_2 = \varphi_1 \pm \alpha$ smerni kot premice $l_2$.

Z uvedbo neskončnih točk in neskončne premice se v merskih dvodimenzionalnih nalogah (praktični primer uporabe prikazan v zunanjem urezu) izognemo računanju dolžin in smernih kotov. Koordinate neznane točke vedno določimo kot presečišče dveh premic, ki jih določat dani točki ter točki v neskončnosti.

**References**


